arXiv:math/9812043v2 [math.FA] 13 Oct 1999

Determinants of Airy Operators and Applications to Random Matrices

Estelle L. Basor*
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407, USA

Harold Widom†
Department of Mathematics
University of California
Santa Cruz, CA 95064, USA

Abstract

The purpose of this paper is to describe asymptotic formulas for determinants of certain operators that are analogues of Wiener-Hopf operators. The determinant formulas yield information about the distribution functions for certain random variables that arise in random matrix theory when one rescales at "the edge of the spectrum".

1 Introduction

This paper is concerned with the asymptotics of Fredholm determinants of operators that arise naturally in random matrix theory and are similar in many ways to finite Wiener-Hopf operators. The operators, denoted by $A_\alpha(f)$, are integral operators on $L^2(\mathbf{R})$ with kernel given by

$$f(x/\alpha)\int_0^\infty A(x+z)A(z+y)dz \tag{1}$$

where

$$A(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it^3/3}e^{itx}dt,$$

and $f\in L^\infty(\mathbf{R})$. The function $A(x)$ is the Airy function, generally denoted by $\mathrm{Ai}(x)$, and for this reason we call our operators $A_\alpha(f)$ *Airy operators*. We will refer to the function f as the *symbol* of the Airy operator.

If the term $\int_0^\infty A(x+z)A(y+z)dz$ in (1) is replaced by the sine kernel

$$\frac{\sin\pi(x-y)}{x-y},$$

*Supported in part by NSF Grant DMS-9623278.
†Supported in part by NSF Grant DMS-9732687.

the resulting operator has the same Fredholm determinant as a finite Wiener-Hopf operator, whose asymptotics are very well known [6]. The similarities with the Wiener-Hopf (i.e., sine kernel) case become less surprising after the observation that

$$\int_0^\infty A(x+z)A(y+z)dz = \frac{A(x)A'(y) - A(y)A'(x)}{x-y}. \tag{2}$$

The proof of this as well as many other facts about the above kernel can be found in [8]. We shall use the fact that $A(x)$ is rapidly decreasing at $+\infty$ and $O(|x|^{-1/4})$ at $-\infty$. For the complete asymptotics of the Airy function, we refer the reader to [5].

Under appropriate conditions $A_\alpha(f)$ is a trace class operator, and thus the Fredholm determinant

$$\det(I + A_\alpha(f))$$

is defined. The main goal of the paper is to compute the asymptotics of this determinant as $\alpha \to \infty$.

The motivation for finding an asymptotic formula for the Fredholm determinant comes from random matrix theory, in studying so-called *linear statistics*, which are certain functions of the eigenvalues of random matrices. After a rescaling at "the edge of the spectrum" their characteristic functions become Fredholm determinants of our Airy operators. For general information about random matrices, we refer the reader to [7]. For information about the connection of random matrices, characteristic functions and the Airy operators we refer the reader to [2] and [8].

The paper is organized as follows. In the second section we derive the basic properties of $A_\alpha(f)$ and related operators. In the third section we prove through a series of lemmas that for appropriate functions f and F

$$\lim_{\alpha\to\infty} \operatorname{tr}\left[F(A_\alpha(f)) - A_\alpha(F \circ f)\right] = \operatorname{tr}\left[F(W(g)) - W(F \circ g)\right],$$

where $W(g)$ is the Wiener-Hopf operator with symbol $g(x) = f(-x^2)$. (The precise definition of $W(g)$ will be given at the end of the next section.) The trace of the second operator on the left is easy to compute asymptotically. Taking $F(z) = \log(1+z)$ and using the known formula for the trace on the right side, we find that the Fredholm determinant is given asymptotically as $\alpha \to \infty$ by

$$\det\left(I + A_\alpha(f)\right) = \exp\left\{c_1\,\alpha^{3/2} + c_2 + o(1)\right\}, \tag{3}$$

where

$$c_1 = \frac{1}{\pi}\int_0^\infty \sqrt{x}\,\log(1 + f(-x))\,dx,$$

$$c_2 = \frac{1}{2}\int_0^\infty x\,(G(x))^2\,dx,$$

and

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} \, \log\left(1 + f(-y^2)\right) dy.$$

This is proved under the assumption that f is a Schwartz function (although we could get by with much less) and $1 + f(x) \neq 0$ for $x \leq 0$.

This formula bears a strong resemblance to the corresponding asymptotic formula in the classical finite Wiener-Hopf case. The most notable difference is the power $\alpha^{3/2}$ in the first term of the asymptotics.

In the last section we describe the implications of formula (3) for random matrix theory. The formula, as in the analogous Wiener-Hopf or Bessel kernel case (see [2] for details), proves that the distribution functions for certain linear statistics, now scaled at the edge of the spectrum, are asymptotically Gaussian. The recurrence of the Gaussian distribution highlights the universality seen again and again in random matrix models.

2 Basic properties of the Airy operator

We begin by defining the *Airy transform* $\mathcal{A}$. For $g \in L^2(\mathbf{R})$ we define $\mathcal{A}(g)$ by the formula

$$\mathcal{A}(g) = \mathcal{F}^{-1} M_h \mathcal{F}^{-1}(g),$$

where $\mathcal{F}(g)(x)$ is the Fourier transform of g given by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-ixt} dt,$$

$\mathcal{F}^{-1}$ is the inverse transform and M_h denotes multiplication by the function $h(t) = e^{it^3/3}$. We will also use the standard notation $\hat{g}$ and $\check{g}$ for the Fourier transform and inverse transforms respectively. Observe that for $g \in L^1 \cap L^2$ we have $\mathcal{A}(g)(x) = \int_{-\infty}^{\infty} A(x+y) g(y) dy$ and, just as in the Fourier transform case, $\mathcal{A}(g)$ is the L^2 limit of $\int_{-B}^{B} A(x+y) g(y) dy$ as $B \to \infty$ for all $g \in L^2$.

Lemma 2.1 *The Airy transform is unitary on L^2 and satisfies $\mathcal{A}^{-1} = \mathcal{A}$.*

Proof. Clearly $\mathcal{A}$ is unitary since $\mathcal{F}$ and M_h are. The Fourier inversion formula says that $\mathcal{F}^{-1} = \mathcal{J}\,\mathcal{F} = \mathcal{F}\,\mathcal{J}$, where $\mathcal{J} g(x) = g(-x)$. It follows that

$$\mathcal{A} = \mathcal{F}\,\mathcal{J}\,M_h\,\mathcal{J}\,\mathcal{F} = \mathcal{F}\,M_{h^{-1}}\,\mathcal{F} = \mathcal{A}^{-1}. \qquad \square$$

Given the above definition of the Airy transform, we see that the Airy operator defined by (1) is alternatively defined as the operator $M_{f_\alpha} \mathcal{A} P \mathcal{A}$, where P is multiplication by $\chi_{\mathbf{R}^+}$ and $f_\alpha(x) = f(x/\alpha)$. It is this representation of the operator which we will use. For appropriate f this operator in turn will have the same Fredholm determinant as $\mathcal{A} M_{f_\alpha} \mathcal{A} P$. We next derive a representation for the kernel of $\mathcal{A} M_{f_\alpha} \mathcal{A}$ for a class of functions f.

Lemma 2.2 *Suppose that f is the inverse Fourier transform of a finite measure μ,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x}\, d\mu(\xi).$$

Then the kernel of the operator $\mathcal{A} M_f \mathcal{A}$ is given by the formula

$$\frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\xi^3/12}}{\sqrt{i\xi+0}}\, e^{-i(x+y)\xi/2}\ e^{i(x-y)^2/4\xi}\, d\mu(\xi), \tag{4}$$

where $\sqrt{i\xi+0}$ is defined by taking $\arg(i\xi+0)$ equal to $\pi/2$ when $\xi > 0$ and $-\pi/2$ when $\xi < 0$.

Proof. Consider first the case where μ is a unit mass at the point η, and let K_η be the operator with the corresponding kernel (4). Then $f(x)$ is the function $e_\eta(x) = e^{i\eta x}/\sqrt{2\pi}$. By Lemma 2.1 we see that we have to show

$$K_\eta = \mathcal{F}^{-1}\, M_h\, \mathcal{F}^{-1}\, M_{e_\eta}\, \mathcal{F}\, M_{h^{-1}}\, \mathcal{F},$$

or equivalently

$$K_\eta\, \mathcal{F}^{-1} = \mathcal{F}^{-1}\, M_h\, \mathcal{F}^{-1}\, M_{e_\eta}\, \mathcal{F}\, M_{h^{-1}}.$$

Let us compute both sides applied to a function in L^2.

Notice first that $\mathcal{F}^{-1}\, M_{e_\eta}\, \mathcal{F}$ takes a function $\varphi(\xi)$ into $\varphi(\xi+\eta)/\sqrt{2\pi}$. Therefore $M_h\, \mathcal{F}^{-1}\, M_{e_\eta}\, \mathcal{F}\, M_{h^{-1}}$ takes $\varphi(\xi)$ into

$$\frac{1}{\sqrt{2\pi}}\, e^{i\xi^3/3}\, e^{-i(\xi+\eta)^3/3}\, \varphi(\xi+\eta) = \frac{1}{\sqrt{2\pi}}\, e^{-i(\xi^2\eta+\xi\eta^2+\eta^3/3)}\, \varphi(\xi+\eta).$$

Hence

$$\begin{aligned}\mathcal{F}^{-1}\, M_h\, \mathcal{F}^{-1}\, M_{e_\eta}\, \mathcal{F}\, M_{h^{-1}}\, \varphi(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x}\, e^{-i(\xi^2\eta+\xi\eta^2+\eta^3/3)}\, \varphi(\xi+\eta)\, d\xi \\ &= \frac{e^{-i(\eta x+\eta^3/3)}}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x}\, e^{-i(\xi^2\eta-\xi\eta^2)}\, \varphi(\xi)\, d\xi. \end{aligned} \tag{5}$$

On the other hand, we have

$$K_\eta\, \mathcal{F}^{-1}\, \varphi(\xi) = \frac{1}{\sqrt{8\pi}} \frac{e^{-i\eta^3/12}}{\sqrt{i\eta+0}}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\xi)\, d\xi \int_{-\infty}^{\infty} e^{i\xi y}\, e^{-i(x+y)\eta/2}\, e^{i(x-y)^2/4\eta}\, dy.$$

(If φ is a Schwartz function, for example, the interchange of order of integration this involves can be justified by integration by parts, and it suffices to show our two operators agree when applied to Schwartz functions.) The inner integral is easily computed and found to equal

$$2\sqrt{\pi}\, \sqrt{i\eta+0}\, e^{i(\xi-\eta)x}\, e^{-i(\xi^2\eta-\xi\eta^2)}\, e^{-i\eta^3/4}.$$

Thus we see that $K_\eta\,\mathcal{F}^{-1}\,\varphi(\xi)$ is equal to the right side of (5).

The lemma is established for the special case of a unit point mass, and so for any linear combination of these. To establish the general result we approximate our given μ by a sequence $\{\mu_n\}$ of linear combinations of point masses such that the measures μ_n are uniformly bounded and $\int\varphi(\xi)\,d\mu_n(\xi)\to\int\varphi(\xi)\,d\mu(\xi)$ for any function φ which is bounded and continuous. Then the corresponding functions f_n converge boundedly and pointwise to f and so the operators $\mathcal{A}M_{f_n}\mathcal{A}$ converge strongly to $\mathcal{A}M_f\mathcal{A}$. For the corresponding operators K_n, it is easy to see that for Schwartz functions g_1 and g_2 we have $(K_n\,g_1,\,g_2)\to(K\,g_1,\,g_2)$, so $K_n\to K$ weakly. Hence, since $\mathcal{A}M_{f_n}\mathcal{A}=K_n$ for each n, we have $\mathcal{A}M_f\mathcal{A}=K$. □

To end this section we recall the definition of a Wiener-Hopf operator and certain of its properties. For a function $g\in L^\infty(\mathbf{R})$ the operator $W(g)$ on $L^2(\mathbf{R}^+)$ (which we identify with the functions in $L^2(\mathbf{R})$ which vanish on $\mathbf{R}^-$) is defined by

$$W(g)=P\mathcal{F}^{-1}M_g\mathcal{F}P.$$

This is the *Wiener-Hopf operator with symbol g*. Notice the analogy with the operators $P\mathcal{A}M_f\mathcal{A}P$. The fact that there is more than just an analogy will become apparent in the next section. One often sees a Wiener-Hopf operator defined as an operator on $L^2(\mathbf{R}^+)$ with kernel of the form $k(x-y)$ where $k\in L^1(\mathbf{R})$. This operator is equal to $W(g)$ with $g(x)=\int e^{-ixu}\,k(u)\,du$.

We state as a lemma two basic facts about Wiener-Hopf operators.

Lemma 2.3 *a) The spectrum of $W(g)$ is contained in the convex hull of the essential range of g.*

b) If g is continuous and $g(\pm\infty)=0$ then λ is not in the spectrum of $W(g)$ if and only if $\lambda\neq 0,\ g-\lambda\neq 0$ and

$$i(g-\lambda):=\frac{1}{2\pi}\underset{-\infty<x<\infty}{\Delta}\arg\left(g(x)-\lambda\right)=0.$$

3 Trace norm estimates and the Airy limit theorem

We assume from now on that f is a Schwartz function. The reader can verify that this requirement is too restrictive and can, for example, be replaced by a weighted space condition. However, assuming that f is a Schwartz function simplifies the proofs and increases the clarity of the arguments.

Recall that the Airy operator $A_\alpha(f)$ equals $M_{f_\alpha}\mathcal{A}P\mathcal{A}$ and is thus similar to $\mathcal{A}M_{f_\alpha}\mathcal{A}P$, which in turn is to unitarily equivalent to the operator $U^{-1}\mathcal{A}M_{f_\alpha}\mathcal{A}PU$ where U is the unitary operator defined by $Ug(x)=\alpha^{1/4}g(x\sqrt{\alpha})$. Note that U commutes with P. This operator will act as a replacement for the Airy operator in the final computations. The next lemma, which involves a modification of the above operator, will be important in those computations.

Lemma 3.1 *The operator $U^{-1}(I-P)\mathcal{A}M_{f_\alpha}\mathcal{A}PU$ converges in the trace norm to the operator with kernel*

$$\frac{1}{\sqrt{8}\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{i(x-y)^2/4\xi}\,d\xi\,\chi_{\mathbf{R}^-}(x)\,\chi_{\mathbf{R}^+}(y)$$

as $\alpha\to\infty$.

Proof. By Lemma 2.2 the kernel of the operator $(I-P)\mathcal{A}M_{f_\alpha}\mathcal{A}P$ is given by the formula

$$\frac{1}{\sqrt{8}\pi}\int_{-\infty}^{\infty}\frac{\alpha\,\hat{f}(\alpha\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12}\,e^{-i(x+y)\xi/2}\,e^{i(x-y)^2/4\xi}\,d\xi\,\chi_{\mathbf{R}^-}(x)\chi_{\mathbf{R}^+}(y),$$

and thus the kernel of $U^{-1}(I-P)\mathcal{A}M_{f_\alpha}\mathcal{A}PU$ is given by

$$\frac{1}{\sqrt{8}\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,e^{-i(x+y)\xi/2\alpha^{3/2}}\,e^{i(x-y)^2/4\xi}\,d\xi\,\chi_{\mathbf{R}^-}(x)\chi_{\mathbf{R}^+}(y).$$

Changing x to $-x$ in the kernel for convenience gives

$$\frac{1}{\sqrt{8}\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,e^{i(x-y)\xi/2\alpha^{3/2}}\,e^{i(x+y)^2/4\xi}\,d\xi\,\chi_{\mathbf{R}^+}(x)\chi_{\mathbf{R}^+}(y).$$

We shall show that replacing by 1 each of the two factors in the integrand which involve α leads to an error which is the kernel of an operator having trace norm $o(1)$. We will use the general fact that the trace norm of an operator with kernel $K(x,y)$, where y is confined to a set J, is at most $\|\,K\,\|_2+|J|\;\|\,\partial K/\partial y\,\|_2$, where the norms are Hilbert-Schmidt norms.

We first look at the error kernel arising from the replacement $e^{-i(x-y)\xi/2\alpha^{3/3}}\to 1$, which is

$$\frac{1}{\sqrt{8}\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,(e^{i(x-y)\xi/2\alpha^{3/2}}-1)\,e^{i(x+y)^2/4\xi}\,d\xi\,\chi_{\mathbf{R}^+}(x)\chi_{\mathbf{R}^+}(y).$$

This we call $K(x,y)$ and find bounds on K and $\partial K/\partial y$.

Clearly $K(x,y)=O(|x-y|/\alpha^{3/2})=O(w/\alpha^{3/2})$, where $w=x+y$. We use this estimate for $w\le 1$. To get a better estimate for $w\ge 1$ we write the kernel as a constant times

$$\frac{1}{w^2}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,(e^{i(x-y)\xi/2\alpha^{3/2}}-1)\,\xi^2\frac{d}{d\xi}e^{i(x+y)^2/4\xi}\,d\xi$$

and integrate by parts to obtain a constant times

$$\frac{1}{w^2}\int_{-\infty}^{\infty}\frac{d}{d\xi}\Big[\xi^2\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,(e^{i(x-y)\xi/2\alpha^{3/2}}-1)\Big]\,e^{i(x+y)^2/4\xi}\,d\xi.$$

Of course we apply the product rule. Differentiating the various factors in the brackets leads to an extra factor α^{-3} or $w\,\alpha^{-3/2}$, aside from the external factor $1/w^2$. If $w\ge 1$ we

see therefore that integration by parts yields a factor $\alpha^{-3/2} < w >^{-1} = \alpha^{-3/2}(1+w^2)^{-1/2}$ assuming that of course f is in a Schwartz function. Integrating by parts once more leads to a factor of $\alpha^{-3} < w >^{-2}$. Thus for all positive x and y ($w \geq 1$ or $w \leq 1$) we see that our kernel satisfies $K(x,y) = O(\alpha^{-3/2} < w >^{-2})$.

We also have to estimate $\partial K(x,y)/\partial y$. If we differentiate

$$\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi + 0}}\, e^{-i\xi^3/12\alpha^3}\,(e^{i(x-y)\xi/2\alpha^{3/2}} - 1)\, e^{i(x+y)^2/4\xi}\, d\xi$$

we are left with two integrals. In one integral we get an extra $\xi/\alpha^{3/2}$ in the integrand, and the factor $e^{i(x-y)\xi/2\alpha^{3/2}} - 1$ is replaced by $e^{i(x-y)\xi/2\alpha^{3/2}}$. As before this can be seen to be $O(\alpha^{-3/2} < w >^{-2})$. In the other integral we get an extra w/ξ which changes the factor $e^{i(x-y)\xi/2\alpha^{3/2}} - 1$ to

$$w\frac{e^{i(x-y)\xi/2\alpha^{3/2}} - 1}{\xi}.$$

Here after three integration by parts we arrive at an estimate of $O(\alpha^{-3/2} < w >^{-2})$.

We have shown that

$$K(x,y), \quad \frac{\partial}{\partial y}K(x,y) = O\left(\frac{\alpha^{-3/2}}{1+x^2+y^2}\right).$$

If we use the general trace norm estimate stated above, taking $J = (k,\ k+1)$ for $k = 0,\ 1, \cdots$ and adding, we find that the trace norm of the error operator is $O(\alpha^{-3/2})$.

If we consider the error due to the replacement $e^{-i\xi^3/12\alpha^3} \to 1$ the argument is essentially the same and we find a bound for the trace norm of the resulting kernel of $O(\alpha^{-3})$. This completes the proof. □

Here and below we shall use the notations $O_1(\cdot)$ resp. $o_1(\cdot)$ to denote families of operators depending on the parameter α whose trace norms are $O(\cdot)$ resp. $o(\cdot)$.

Lemma 3.2 . *We have $P\mathcal{A}M_{f_\alpha}\mathcal{A}P = O_1(\alpha^{3/2})$ in general and $P\mathcal{A}M_{f_\alpha}\mathcal{A}P = o_1(1)$ if f vanishes on $\mathbf{R}^-$.*

Proof. The kernel of our operator on $L^2(\mathbf{R}^+)$ equals

$$\int_{-\infty}^{\infty} f(z/\alpha)\, A(x+z)\, A(y+z)\, dz.$$

For fixed z the kernel $f(z/\alpha)\, A(x+z)\, A(y+z)$ is a separable rank one kernel. To compute its trace norm, observe that by the estimates on the Airy function we have

$$\int_0^{\infty} A(x+z)^2\, dx = \begin{cases} O(e^{-z}), & z > 0, \\ O(< z >^{1/2}), & z < 0. \end{cases}$$

It follows that the trace norm of our operator is at most a constant times

$$\int_{-\infty}^{0} |f(z/\alpha)| < z >^{1/2} \, dz + \int_{0}^{\infty} |f(z/\alpha)| \, e^{-z} \, dz,$$

and the assertions of the lemma follow easily. $\square$

Corollary 3.3 $A_\alpha(f)$ *is a trace class operator.*

Proof. After the replacement $x \to -x$ the kernel in the statement of Lemma 3.1 becomes a Hankel operator with smooth kernel and thus is well-known to be trace class. The lemma implies that $U^{-1}(I-P)\mathcal{A}M_{f_\alpha}\mathcal{A}PU$ is trace class. Thus $(I-P)\mathcal{A}M_{f_\alpha}\mathcal{A}P$ is trace class. Lemma 3.2 tells us that $P\mathcal{A}M_{f_\alpha}\mathcal{A}P$ is trace class. Hence so is $\mathcal{A}M_{f_\alpha}\mathcal{A}P$, and $A_\alpha(f) = M_{f_\alpha}\mathcal{A}P\mathcal{A}$ is unitarily equivalent to this. $\square$

We remark that this argument could have been made much earlier. However it would have involved the same sort of estimates as in the proof of Lemma 3.1 and there was no point in doing this twice.

The operator with kernel

$$\frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi + 0}} e^{i(x-y)^2/4\xi} d\xi$$

is a convolution operator, its kernel is a function of $x - y$. We will denote it by K_f. Thus PK_fP is a Wiener-Hopf operator with symbol

$$g(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi + 0}} e^{iu^2/4\xi} e^{-ixu} d\xi du.$$

Now

$$\int_{-\infty}^{\infty} e^{i(\frac{u^2}{4\xi} - xu)} \, du = \sqrt{4\pi}\sqrt{i\xi + 0} \, e^{-ix^2\xi},$$

and therefore our symbol is given by

$$g(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-ix^2\xi} d\xi = f(-x^2).$$

Thus $PK_fP = W(g)$, and the connection to Wiener-Hopf operators is now apparent.

Lemma 3.1 told us that the operator $U^{-1}(I-P)\mathcal{A}M_{f_\alpha}\mathcal{A}PU$ converges in the trace norm to $(I-P)W(g)P$. The next lemma concerns the strong convergence of the operator

$$B_\alpha(f) = U^{-1}\mathcal{A}M_{f_\alpha}\mathcal{A}U.$$

This is the last technical lemma before we can put the pieces together.

Lemma 3.4 *The operator $B_\alpha(f)$ converges strongly to K_f as $\alpha \to \infty$.*

Proof. We have to show that for any $\varphi \in L^2(\mathbf{R})$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, e^{-i\xi^3/12\alpha^3}\, e^{-i(x+y)\xi/2\alpha^{3/2}}\, e^{i(x-y)^2/4\xi}\, \varphi(y)\, dy\, d\xi$$

$$\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, e^{i(x-y)^2/4\xi}\, \varphi(y)\, dy\, d\xi.$$

in $L^2(\mathbf{R})$. We can restrict ourselves to a dense subset of φs since the $B_\alpha(f)$ have uniformly bounded norms.

Write the double integral on the left as

$$\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, e^{-i\xi^3/12\alpha^3}\, e^{-ix\xi/2\alpha^{3/2}} \int_{-\infty}^{\infty} e^{-iy\xi/2\alpha^{3/2}}\, e^{i(x-y)^2/4\xi}\, \varphi(y)\, dy\, d\xi.$$

This minus its purported L^2 limit equals the sum of the two error integrals

$$\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, e^{-i\xi^3/12\alpha^3}\, e^{-ix\xi/2\alpha^{3/2}} \int_{-\infty}^{\infty} (e^{-iy\xi/2\alpha^{3/2}}-1)\, e^{i(x-y)^2/4\xi}\, \varphi(y)\, dy\, d\xi \tag{6}$$

and

$$\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, (e^{-i\xi^3/12\alpha^3}\, e^{-ix\xi/2\alpha^{3/2}}-1) \int_{-\infty}^{\infty} e^{i(x-y)^2/4\xi}\, \varphi(y)\, dy\, d\xi. \tag{7}$$

The operator with kernel $e^{i(x-y)^2/\xi}$ is unitarily equivalent to, and therefore has the same norm as, the operator with kernel $|\xi|^{1/2}\, e^{i(x-y)^2}$. Thus it has norm $O(|\xi|^{1/2})$. The function

$$(e^{-iy\xi/\alpha^{3/2}}-1)\, \varphi(y)$$

has norm $O(\xi/\alpha^{3/2})$, assuming as we may that $y\, \varphi(y) \in L^2$, and it follows that the inner integral in (6) has norm $O(|\xi|^{3/2}/\alpha^{3/2})$. Hence (6) itself has norm at most $O(1/\alpha^{3/2})$.

As for (7), the inner integral equals a function $\psi_\xi(x)$ whose L^2 norm is $O(|\xi|^{1/2})$. Write (7) as the sum

$$\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, (e^{-i\xi^3/12\alpha^3}-1)\, e^{-ix\xi/2\alpha^{3/2}}\, \psi_\xi(x)\, d\xi + \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\, (e^{-ix\xi/2\alpha^{3/2}}-1)\, \psi_\xi(x)\, d\xi. \tag{8}$$

The L^2 norm of the function $(e^{-i\xi^3/12\alpha^3}-1)\, e^{-ix\xi/2\alpha^{3/2}}\, \psi_\xi(x)$ is $O(|\xi|^{7/2}/\alpha^3)$ and so the first integral in (7) is a function whose norm is $O(1/\alpha^3)$. As for the second integral, observe that

$$\|\, (e^{-ix\xi/\alpha}-1)\, \psi_\xi(x)\, \|$$

is $O(\xi)$ for all ξ and α and tends to 0 as $\alpha \to \infty$ for each ξ (by the dominated convergence theorem). Hence the integral obtained by taking norm under the integral sign in the second integral tends to 0 as $\alpha \to \infty$, again by the dominated convergence theorem. This establishes the claimed strong convergence. $\square$

Now we are ready to begin the final steps in proving (3). The operator $(A_\alpha(f))^n = (M_{f_\alpha}\mathcal{A}P\mathcal{A})^n$ has the same trace as $(P\mathcal{A}M_{f_\alpha}\mathcal{A}P)^n$ which in turn has the same trace as $(PU^{-1}\mathcal{A}M_{f_\alpha}\mathcal{A}UP)^n = (PB_\alpha(f)P)^n$. In fact for any analytic function F defined on the neighborhood of the spectra of both operators and satisfying $F(0) = 0$ we have

$$\operatorname{tr} F(A_\alpha(f)) = \operatorname{tr} F(PB_\alpha(f)P).$$

It is the asymptotics of this last trace we shall compute. We think of our operators as acting on $L^2(\mathbf{R}^+)$.

In the following two lemmas λ will be in the resolvent set of the Wiener-Hopf operator $PK_fP = W(g)$. By Lemma 2.3b this implies that λ is not in the range of $g(x) = f(-x^2)$, so $f^{-1}(\{\lambda\})$ is a compact subset of $(0, \infty)$. We can find a Schwartz function $\tilde{f}$ which never takes the value λ and which equals f outside some larger compact subset of $(0, \infty)$, and we can find one $\tilde{f}$ which serves for all λ in any given compact subset of the resolvent set of $W(g)$. This will be our notation in what follows.

Lemma 3.5 *Let λ be in the resolvent set of $PK_fP = W(g)$. Then λ is also in the resolvent set of $PB_\alpha(f)P$ for sufficiently large α and the inverses have uniformly bounded norms for λ lying in any given compact subset of the resolvent set.*

Proof. Observe that for Schwartz functions f_1 and f_2, since $B_\alpha(f_1)\,B_\alpha(f_2) = B_\alpha(f_1\,f_2)$,

$$PB_\alpha(f_1)PB_\alpha(f_2)BP - PB_\alpha(f_1\,f_2)BP = PB_\alpha(f_1)(P-I)B_\alpha(f_2)BP$$

$$\to PK_{f_1}(P-I)K_{f_2}P = PK_{f_1}PK_{f_2}P - PK_{f_1f_2}P = W(g_1)\,W(g_2) - W(g_1\,g_2) \qquad (9)$$

in trace norm since $(I-P)B_\alpha(f_2)BP \to (I-P)K_{f_2}P$ in trace norm by Lemma 3.1 and $PB_\alpha(f_1) \to PK_{f_1}$ strongly by Lemma 3.4. This also holds if the f_i are constants plus Schwartz functions.

We take $f_1 = f - \lambda$ and $f_2 = (\tilde{f} - \lambda)^{-1}$. Observe that the "$g$" corresponding to f_2 is $(g-\lambda)^{-1}$, so in this case the relation (9) reads

$$PB_\alpha(f-\lambda)PB_\alpha((\tilde{f}-\lambda)^{-1})P - PB_\alpha((f-\lambda)\,(\tilde{f}-\lambda)^{-1})P \to W(g-\lambda)\,W((g-\lambda)^{-1}) - P.$$

Now

$$PB_\alpha((f-\lambda)\,(\tilde{f}-\lambda)^{-1})P = P + PB_\alpha((f-\lambda)\,(\tilde{f}-\lambda)^{-1} - 1)P = P + o_1(1),$$

by Lemma 3.2. We conclude that

$$PB_\alpha(f-\lambda)PB_\alpha((\tilde{f}-\lambda)^{-1})P = W(g-\lambda)\,W((g-\lambda)^{-1}) + o_1(1). \tag{10}$$

The analogous formula holds when $f-\lambda$ and $(\tilde{f}-\lambda)^{-1}$ are interchanged. The Wiener-Hopf operators $W(g-\lambda)$ and $W((g-\lambda)^{-1})$ are invertible, the first by assumption and the second since $\operatorname{ind}(g-\lambda)^{-1} = -\operatorname{ind}(g-\lambda) = 0$. The norms of the inverses are bounded uniformly in α and λ lying in a compact set and the operators $PB_\alpha((\tilde{f}-\lambda)^{-1})P$ are uniformly bounded. This completes the proof. $\square$

Lemma 3.6 *Suppose F is analytic in a neighborhood of the spectrum of $W(g)$. Then we have as $\alpha \to \infty$*

$$F(PB_\alpha(f)P) - PB_\alpha(F\circ\tilde{f})P = F(W(g)) - W(F\circ g) + o_1(1). \tag{11}$$

Proof. By Lemma 3.5 $PB_\alpha(f-\lambda)P$ is invertible for sufficiently large α with uniformly bounded norm for λ lying in a compact set in the resolvent set of $W(g)$. Let λ also be in the domain of F. Then

$$(PB_\alpha(f-\lambda)P)^{-1} - PB_\alpha((\tilde{f}-\lambda)^{-1})P = (PB_\alpha(f-\lambda)P)^{-1}\,[I - W(g-\lambda)\,W((g-\lambda)^{-1})] + o_1(1)$$

$$= W(g-\lambda)^{-1}\,[I - W(g-\lambda)\,W((g-\lambda)^{-1})] + o_1(1).$$

The first equality follows from (10) and the uniformity of the norms of the inverses. The second equality uses the strong convergence of $(PB_\alpha(f-\lambda)P)^{-1}$ to $W(g-\lambda)^{-1}$ and the fact that $I - W(g-\lambda)\,W((g-\lambda)^{-1})$ is trace class. Thus

$$(PB_\alpha(f-\lambda)P)^{-1} - PB_\alpha((\tilde{f}-\lambda)^{-1})P = W(g-\lambda)^{-1} - W((g-\lambda)^{-1}) + o_1(1).$$

Multiplying by $F(\lambda)$ and integrating over an appropriate contour gives

$$F(PB_\alpha(f)P) - PB_\alpha(F\circ\tilde{f})P = F(W(g)) - W(F\circ g) + o_1(1)$$

for any F analytic in a neighborhood of the spectrum of $W(g)$. $\square$

We will be interested in the trace of the first operator on the left side of (11). The next lemma will tell us the trace of the second operator.

Lemma 3.7 *For any Schwartz function f we have*

$$\operatorname{tr} PB_\alpha(f)P = \frac{\alpha^{3/2}}{\pi}\int_0^\infty \sqrt{x}\,f(-x)\,dx + o(1).$$

Proof. The kernel of $PB_\alpha(f)P$ equals

$$\frac{1}{\sqrt{8}\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,e^{-i(x+y)\xi/2\alpha^{3/2}}\,e^{i(x-y)^2/4\xi}\,d\xi\,\chi_{\mathbf{R}^+}(x)\chi_{\mathbf{R}^+}(y),$$

and thus

$$\operatorname{tr} PB(f)P=\frac{1}{\sqrt{8}\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,e^{-ix\xi/\alpha^{3/2}}\,d\xi\,dx.$$

We write this as

$$\frac{\alpha^{3/2}}{\sqrt{8}\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-i\xi^3/12\alpha^3}\,e^{-ix\xi}\,d\xi\,dx,$$

and then replace the term $e^{-i\xi^3/12\alpha^3}$ by 1 just as in Lemma 3.1 to find that the trace is given by

$$\frac{\alpha^{3/2}}{\sqrt{8}\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\sqrt{i\xi+0}}\,e^{-ix\xi}\,d\xi\,dx+O(\alpha^{-3/2}).$$

We can write this in a more familiar form by replacing the term $1/\sqrt{i\xi+0}$ in the above integral with $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-iu^2\xi}du$. Integrating over ξ we find that this equals

$$\frac{\alpha^{3/2}}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(-u^2-x)du\,dx+o(1)$$

or

$$\frac{\alpha^{3/2}}{\pi}\int_0^{\infty}\sqrt{x}\,f(-x)\,dx+o(1)$$

as claimed. □

We now derive our main result on determinants of Airy operators which gives the promised formula (3) for the asymptotics. We assume, as always, that f is a Schwartz function.

Theorem 3.8 *Assume $g(x)=f(-x^2)\neq -1$. Then as $\alpha\to\infty$*

$$\det\left(I+A_\alpha(f)\right)=\exp\left\{c_1\,\alpha^{3/2}+c_2+o(1)\right\},\tag{12}$$

where

$$c_1=\frac{1}{\pi}\int_0^{\infty}\sqrt{x}\,\log(1+f(-x))\,dx,$$

$$c_2=\frac{1}{2}\int_0^{\infty}x\left((\log(1+g))\check{\ }(x)\right)^2dx.$$

Proof. Assume first that $\|g\|_\infty < 1$. Then by Lemma 2.3a the spectrum of $W(g)$ lies in the open unit disc with center 0. Therefore $F(z) = \log(1+z)$ (the branch equal to 0 when $z = 0$) is analytic on the spectrum and so we may apply Lemma 3.6. This and Lemma 3.7 tell us that there is an asymptotic formula of the form (12) where c_1 is as stated and

$$c_2 = \mathrm{tr}\,[\log(I + W(g)) - W(\log(1+g))].$$

It is known that this equals the expression given for c_2 in the statement of the theorem [10].

To remove the restriction on g we introduce a parameter t and we would like to define a family of functions f_t by $1 + f_t = e^{t\,\log(1+f)}$, so that for small enough t our asymptotic formula holds. The problem is that -1 may lie in the range if f, and even if it didn't we might not have $i(1+f) = 0$, which is what we need to define a logarithm which is a Schwartz function. So, as in the preceding lemmas, we introduce a function $\tilde{f}$ which equals f except on a compact subset of $(0, \infty)$ such that $1 + f \neq 0$ and $\mathrm{ind}\,(1+\tilde{f}) = 0$. Then we define f_t by

$$1 + f_t = e^{t\,\log(1+\tilde{f})} + f - \tilde{f}.$$

Of course $f_1 = f$. Moreover, with $g_t(x) = f_t(-x^2)$,

$$1 + g_t = e^{t\,\log(1+g)}$$

for all t. For sufficiently small t we have $\|g_t\|_\infty < 1$ so that our formulas hold.

Observe that $\det(I + A_\alpha(f_t))$ is a family of entire functions of t depending on the parameter α. Suppose we can show that

$$\det(I + A_\alpha(f_t)) = O(e^{t\,c_1\alpha^{3/2}}) \tag{13}$$

for large α uniformly on compact t-sets. Then the limit relation

$$\lim_{\alpha\to\infty} e^{-t\,c_1\alpha^{3/2}}\,\det(I + A_\alpha(f_t)) = e^{t\,c_2},$$

which we know holds for sufficiently small t, will hold for all t and therefore $t = 1$.

To prove (13) we go back to the $PB_\alpha(f)P$ regarded as operators on $L^2(\mathbf{R}^+)$. We have

$$\det(I + A_\alpha(f_t)) = \det\,PB_\alpha(1+f_t)P.$$

Now $i(1+g_t) = 0$ and so $W(1+g_t)$ is invertible by Lemma 2.3b. Therefore by Lemma 3.5 with $F(z) = z^{-1}$ we know that $PB_\alpha(1+f_t)P$ will be invertible if α is large enough. (This will hold for all t in any given compact set.) For these α we have

$$\frac{d}{dt}\log\,\det(I+A_\alpha(f_t)) = \mathrm{tr}\,[(PB_\alpha(1+f_t)P)^{-1}\,\frac{d}{dt}PB_\alpha(1+f_t)P] = \mathrm{tr}\,[(PB_\alpha(1+f_t)P)^{-1}\,PB_\alpha(h_t)P],$$

where

$$h_t = \log(1+\tilde{f})\, e^{t\,\log(1+\tilde{f})}.$$

By Lemma 3.6 with $F(z) = z^{-1}$ we know that

$$(PB_\alpha(1+f_t)P)^{-1} = PB_\alpha((1+f_t)^{-1})P + O_1(1).$$

Also, by (9),

$$PB_\alpha((1+f_t)^{-1}))PB_\alpha(h_t)P = PB_\alpha((1+f_t)^{-1}\, h_t)P + O_1(1),$$

so that we have shown

$$\frac{d}{dt}\log\,\det(I + A_\alpha(f_t)) = \mathrm{tr}\, PB_\alpha((1+f_t)^{-1}\, h_t)P + O(1).$$

But Lemma 3.7 tells us that with an error $o(1)$

$$\mathrm{tr}\, PB_\alpha(1+f_t)^{-1}\, h_t)P = \frac{\alpha^{3/2}}{\pi}\int_0^\infty (1+f_t(-x))^{-1}\, h_t(-x)\,\sqrt{x}\,dx$$

$$= \frac{\alpha^{3/2}}{\pi}\int_0^\infty \sqrt{x}\,\log(1+f(-x))\,dx.$$

because $\tilde{f} = f$ on $\mathbf{R}^-$. Thus,

$$\frac{d}{dt}\log\,\det(I + A_\alpha(f_t)) = \frac{\alpha^{3/2}}{\pi}\int_0^\infty \sqrt{x}\,\log(1+f(-x))\,dx + O(1).$$

Integrating over t from 0 to t and exponentiating gives (13) and completes the proof. $\square$

4 Applications to random matrices

Theorem 3.7 can be applied to find limiting distribution functions for a class of random variables which are functions of the eigenvalues of a random matrix. In many different ensembles of matrices it has been shown that the distribution functions are asymptotically normal [1, 2, 3, 4], and this will be shown also to be the case in our examples. The term *ensemble* refers to the probability density assigned to some space of matrices, and this in turn induces a density on the space of eigenvalues of the matrices. For the Gaussian Unitary Ensemble (GUE) the density on the space of eigenvalues is given by

$$P_N(x_1,\ldots,x_N) = \frac{1}{N!}\det K(x_i,x_j)\,|_{i,j=1}^{N} \tag{14}$$

where

$$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\phi_i(y). \tag{15}$$

and ϕ_i is obtained by orthonormalizing the sequence $\{x^i e^{-x^2/2}\}$ over $\mathbf{R}$. If N is large it is also well known that the density of the eigenvalues is supported on approximately the interval $(-\sqrt{2N}, \sqrt{2N})$. These facts can be found in [7].

The random variables of interest here are ones that are often called *linear statistics* and are of the form

$$\sum_{i=1}^{N} f(\lambda_i/\alpha),$$

where λ_i are the eigenvalues and f is an appropriate function. Our goal is to study these random variables applied to the eigenvalues near the edge of the spectrum and to this end we rescale and replace the sum by

$$\sum_{i=1}^{N} f(2^{1/2}N^{1/6}(\lambda_i - \sqrt{2N})/\alpha).$$

The purpose of the translation by the term $\sqrt{2N}$ is to move to the edge and the factor $2^{1/2}N^{1/6}$ has the effect of making the eigenvalue density of the order 1. Otherwise the eigenvalues "bunch up" or "spread out" and all the results become more or less trivial.

To study the distribution function of this random variable we use its characteristic function, or inverse Fourier transform. This characteristic function is given by

$$\phi_N(s) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{is\sum_{j=1}^{N} f(2^{1/2}N^{1/6}(x_j - \sqrt{2N})/\alpha))}\, P_N(x_1, \ldots, x_N)\, dx_1 \cdots dx_N.$$

It is a general fact that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N}(1 + g(x_i))\, P_N(x_1, \ldots, x_N)\, dx_1 \cdots dx_N = \det(I + gK_N),$$

where g denotes multiplication by $g(x)$ and K_N denotes the operator with kernel $K_N(x,y)$. This can be obtained by expanding out the product in the integrand, using the formula

$$\frac{N!}{(N-n)!} \int \cdots \int P_N(x_1, \ldots, x_n, x_{n+1}, \ldots, x_N)\, dx_{n+1} \cdots dx_N = \det K_N(x_i, x_j)\,\Big|_{i,j=1}^{n}$$

for the n-point correlation function, and then recognizing the resulting sum of multiple integrals as the expansion of the Fredholm determinant. Or it can be obtained by a simpler algebraic device [9]. In our case $1 + g(x) = \exp\{f(2^{1/2}N^{1/6}(x - \sqrt{2N})/\alpha)\}$. If we make the changes of variable

$$x \to \frac{x}{2^{1/2}N^{1/6}} + \sqrt{2N}, \quad y \to \frac{y}{2^{1/2}N^{1/6}} + \sqrt{2N}$$

we find that the characteristic function equals the determinant of I plus the operator with kernel

$$(e^{isf(x/\alpha)}-1)\,\frac{1}{2^{1/2}N^{1/6}}\,K_N(\frac{x}{2^{1/2}N^{1/6}}+\sqrt{2N},\ \frac{y}{2^{1/2}N^{1/6}}+\sqrt{2N}).$$

Now one has the scaling limit

$$\lim_{N\to\infty}\frac{1}{2^{1/2}N^{1/6}}\,K_N(\frac{x}{2^{1/2}N^{1/6}}+\sqrt{2N},\ \frac{y}{2^{1/2}N^{1/6}}+\sqrt{2N})$$
$$=\frac{A(x)A'(y)-A'(x)A(y)}{x-y},$$

precisely the Airy kernel. Thus we see that the large N limit of the characteristic function equals $\phi(s)=\det\left(I+A_\alpha(h)\right)$ where $h(x)=e^{isf(x)}-1$. Our asymptotic formula yields

$$\phi(s)=\exp\left\{\frac{is\alpha^{3/2}}{\pi}\int_0^\infty\sqrt{x}f(-x)\,dx-\frac{s^2}{2}\int_0^\infty x\,(\check{g}(x))^2\,dx+o(1)\right\},$$

where as before $g(x)=f(-x^2)$.

Notice that the limiting characteristic function is quadratic in s and hence the distribution is asymptotically normal. Of course this is not surprising since this occurs for other matrix ensembles and other scaling limits. Notice, though, that in this case the mean and variance of the limiting distribution only depend on the negative values of the argument of the original f. This is a reflection of the fact that the Airy function goes rapidly to zero for positive values and oscillates and tends to zero slowly for negative values of the argument. A question left to the future is how the asymptotics of functions $A(x)$ and $B(x)$ in a kernel of the form

$$\frac{A(x)B(y)-A(y)B(x)}{x-y}$$

affect the asymptotics of the corresponding distribution functions.

References

[1] T. H. Baker, P. J. Forrester. – Finite N Fluctuations Formulas for Random Matrices, *J. Stat. Phys.* 88 (1997), 1371–1386.

[2] E. L. Basor. – Distribution Functions for Random Variables for Ensembles of Positive Hermitian Matrices, *Comm. Math. Phys.* 188 (1997), 327–350.

[3] E. L. Basor, C. A. Tracy. – Variance calculations and the Bessel kernel, *J. Stat. Phys.* 73 (1993), 415–421.

[4] C. W. J. Beenakker. – Universality in the random-matrix theory of quantum transport, *Phys. Rev. Letts.* 70 (1993), 1155-1158.

[5] A. Erdélyi (ed.) – *Higher transcendental functions* Vol.II, McGraw-Hill, New York, 1953

[6] M. Kac.–Toeplitz matrices, translation kernels, and a related problem in probability theory, *Duke Math. J.* 21 (1954), 501-509.

[7] M. L. Mehta. – *Random Matrices*, Academic Press, Rev. and enlarged 2nd ed., San Diego, 1991.

[8] C. A. Tracy, H. Widom. – Level-Spacing Distributions and the Airy Kernel, *Comm. Math. Phys.* 159 (1994), 151-174.

[9] C. A. Tracy, H. Widom. – Correlation functions, cluster functions and spacing distributions for random matrices, *J. Stat. Phys.* to appear. 159 (1994), 151-174.

[10] H. Widom. – A trace formula for Wiener-Hopf operators, *J. Oper. Th.* 8 (1982) 279–298.

[11] E. T. Whittaker, G. N. Watson, – *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, London/New York, 1952.